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Asymptotic Estimation of the Eigenvalues and Eigenfunctions for the Sixth Order Boundary Value Problems

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Abstract

This paper addresses an eigenvalue problem generated by sixth order differential equations with suitable boundary conditions, which contain a spectral parameter. The asymptotic expressions for sixth order linearly independent solutions as well as new asymptotic formulas for the eigenvalues with eigenfunctions of the boundary value problem are obtained.

Introduction

A sixth order linear differential operator is generated by the differential equation and boundary conditions of the form:

$$l(y) = -y^{(6)}(x) + q(x)y(x) = \lambda^6 y(x)\,,\ x \in [0, a] \quad (1)$$

$$U_j\big(y(x)\big) = \begin{cases} y^{(j)}(0) = 0,\ j = 0,1,2 \\ \sum_{k=1}^{6} \big(iw_j\lambda\big)^{k-1} y^{(6-k)}(a, \lambda) = 0\,,\ j = 3,4,5 \end{cases} \quad (2)$$

Where λ is the spectral parameter and $q(x)$ is an arbitrary complex-valued function such that $q(x) \in C^2[0, a]$.

Furthermore, λ satisfies:

$q'(0) = q'(a) = 0\ ,\ \int_0^a q(x)dx = 0$ provided $q(a) \neq 0$.

Many authors have investigated the spectral properties of eigenvalues and eigenfunctions of differential equations (H. Menken [1], K. H. Jwamer [2,3,4], G. A. Auginov [5,6], J. D. Tamarkin [7], etc.). It is worth further note that T. Y. Gadzhieva [9] studied the differential equation of the order "$2n$" and the form $y^{(2n)}(x) + q(x)y(x) = \lambda^{2n}\rho(x)y(x), x \in [0, a]$. Gadzhieva's work considered $\rho(x) \neq 1$, and obtained the asymptotic formulas exclusively for the eigenvalues under these conditions.

The aim of this paper is to find a new expression for the six linearly independent solutions and asymptotic formulas of the eigenvalues and eigenfunctions of equations (1)-(2). To achieve this, the auxiliary results proven in section 2 are prerequisite.

2. Expressions of Fundamental Solutions :

In this section we find a new asymptotic expression for the fundamental solutions of(1).

Theorem 1: If we have the differential equation(1).where,$q(x) \in C^{n-1}[0,a]$, then for $\lambda \in T_0$, where $T_0 = \left\{\lambda : \arg\lambda \in \left[0, \frac{\pi}{8}\right]\right\}$ and $w_k, k = 0:5$ are six root of unity we can find six linearly independent solutions and their derivatives can be expressed as

$y_k^{(s)}(x,\lambda) = (i\lambda w_k)^s e^{i\lambda w_k x}\left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + \frac{A_{6sk}(x)}{\lambda^6} + \cdots + \frac{A_{nsk}(x)}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right]$,

Where

$A_{1sk} = A_{1k}(x), A_{2sk} = A_{2k}(x), A_{3sk} = A_{3k}(x), A_{4sk} = A_{4k}(x), A_{5sk} = A_{5k}(x),$

$A_{6sk} = A_{6k}(x) - \binom{S}{1} iw_k^5 A'_{5k}(x),$

$A_{7sk} = A_{7k}(x) - \binom{S}{1} iw_k^5 A'_{6k}(x) - \binom{S}{2} w_k^4 A''_{5k}(x)$

And so on for $n \geq 7$ we have:

$A_{nsk} = A_{n,k}(x) - \binom{S}{1} iw_k^5 A'_{n-1,k}(x) - \binom{S}{2} w_k^4 A''_{n-2,k}(x) + \binom{S}{3} iw_k^3 A'''_{n-3,k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{n-4,k}(x) - \binom{S}{5} \mathrm{i}w_k A^{(5)}_{n-5,k}(x) + \binom{S}{6} A^{(6)}_{n-6,k}(x).$

And

$A_{0,k}(x) = 1, A_{1,k}(x) = 0, A_{2,k}(x) = 0, A_{3,k}(x) = 0, A_{4,k}(x) = 0, A_{5.k}(x) = -\frac{iw_k}{6}\int_0^x q(t)A_{0,k}(t)dt,$

$A_{6,k}(x) = -\frac{iw_k}{6}\int_0^x \left(-15w_k^4 A''_{5,k}(t) + q(t)A_{1,k}(t)\right)dt,$

$$A_{7,k}(x) = -\frac{iw_k}{6}\int_0^x \left(-15w_k^4 A''_{6,k}(t) + 20iw_k^3 A'''_{5,k}(t) + q(t)A_{2,k}(t)\right)dt,$$

$$A_{8,k}(x) = -\frac{iw_k}{6}\int_0^x \left(-15w_k^4 A''_{7,k}(t) + 20iw_k^3 A'''_{6,k}(t) + 15w_k^2 A^{(4)}_{5,k}(t) + q(t)A_{3,k}(t)\right)dt,$$

And hence for integer , $n \geq 8$:

$$A_{n.k}(x) = -\frac{iw_k}{6}\int_0^x \Big(-15w_k^4 A''_{\mathrm{n}-1,k}(t) + 20iw_k^3 A'''_{\mathrm{n}-2,k}(t) + 15w_k^2 A^{(4)}_{\mathrm{n}-3,k}(t) + A^{(5)}_{\mathrm{n}-4,k}(t) + A^{(6)}_{\mathrm{n}-5,k}(t) + q(t)A_{\mathrm{n}-5,k}(t)\Big)dt,$$

Proof: As we see in [1] the solution of the differential equation can be written in a power series of the form $y_k(x,\lambda) = e^{\lambda\int_0^x \phi_k dt}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$,where $\phi_k(x) = iw_k\sqrt[6]{\rho(x)}$, but in our problem $\rho(x) = 1$,can be written as

$y_k(x,\lambda) = e^{i\lambda w_k x}\sum_{j=0}^{\infty}\frac{A_j(x)}{\lambda^j}$

We try to find $y'_k, y''_k, y'''_k, y_k^{(4)}, y_k^{(5)}, y_k^{(6)}$ and putting in the differential equation, first

$y_k(x,\lambda) = e^{i\lambda w_k x}\left[A_0(x) + \frac{A_1(x)}{\lambda} + \cdots + \frac{A_n(x)}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right]$, (3)

$$y_k'(x,\lambda) = i\lambda w_k e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - iw_k^5 A_1'(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - iw_k^5 A_2'(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - iw_k^5 A_3'(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - iw_k^5 A_{n-1}'(x)\Big) + O\Big(\frac{1}{\lambda^{n+1}}\Big)\Big] \quad (4)$$

$$y_k''(x,\lambda) = (i\lambda w_k)^2 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - 2iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - 2iw_k^5 A_1'(x) - w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - 2iw_k^5 A_2'(x) - w_k^4 A_1''(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - 2iw_k^5 A_3'(x) - w_k^4 A_2''(x)\Big) + \frac{1}{\lambda^5}\Big(A_5(x) - 2iw_k^5 A_4'(x) - w_k^4 A_3''(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - 2iw_k^5 A_{n-1}'(x) - w_k^4 A_{n-2}''(x)\Big) + O\Big(\frac{1}{\lambda^{n+1}}\Big)\Big], \quad (5)$$

$$y_k'''(x,\lambda) = (i\lambda w_k)^3 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - 3iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - 3iw_k^5 A_1'(x) - 3w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - 3iw_k^5 A_2'(x) - 3w_k^4 A_1''(x) + iw_k^3 A_0'''(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - 3iw_k^5 A_3'(x) - 3w_k^4 A_2''(x) + iw_k A_1'''(x)\Big) + \frac{1}{\lambda^5}\Big(A_5(x) - 3iw_k^5 A_4'(x) - 3w_k^4 A_3''(x) + iw_k A_2'''(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - 3iw_k^5 A_{n-1}'(x) - 3w_k^4 A_{n-2}''(x) + iw_k^3 A_{n-3}'''(x)\Big) + O\Big(\frac{1}{\lambda^{n+1}}\Big)\Big] \quad (6)$$

$$y_k^{(4)}(x,\lambda) = (i\lambda w_k)^4 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - 4iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - 4iw_k^5 A_1'(x) - 6w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - 4iw_k^5 A_2'(x) - 6w_k^4 A_1''(x) + 4iw_k^3 A_0'''(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - 4iw_k^5 A_3'(x) - 6w_k^4 A_2''(x) + 4iw_k^3 A_1'''(x) + A_0^{(4)}(x)\Big) + \frac{1}{\lambda^5}\Big(A_5(x) - 4iw_k^5 A_4'(x) - 6w_k^4 A_3''(x) + 4iw_k^3 A_2'''(x) + A_1^{(4)}(x)\Big) + \frac{1}{\lambda^6}\Big(A_6(x) - 4iw_k^5 A_5'(x) - 6w_k^4 A_4''(x) + 4iw_k^3 A_3'''(x) + A_2^{(4)}(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - 4iw_k^5 A_{n-1}'(x) - 6w_k^4 A_{n-2}''(x) + 4iw_k^3 A_{n-3}'''(x) + A_{n-4}^{(4)}(x)\Big) + O\Big(\frac{1}{\lambda^{n+1}}\Big)\Big], \quad (7)$$

$$y_k^{(5)}(x,\lambda) = (i\lambda w_k)^5 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - 5iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - 5iw_k^5 A_1'(x) - 10w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - 5iw_k^5 A_2'(x) - 10w_k^4 A_1''(x) + 10iw_k^3 A_0'''(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - 5iw_k^5 A_3'(x) - 10w_k^4 A_2''(x) + 10iw_k^3 A_1'''(x) + 5w_k^2 A_0^{(4)}(x)\Big) + \frac{1}{\lambda^5}\Big(A_5(x) - 5iw_k^5 A_4'(x) - 10w_k^4 A_3''(x) + 10iw_k^3 A_2'''(x) + 5w_k^2 A_1^{(4)}(x) + A_0^{(5)}(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - 5iw_k^5 A_{n-1}'(x) - 10w_k^4 A_{n-2}''(x) + 10iw_k^3 A_{n-3}'''(x) + 5w_k^2 A_{n-4}^{(4)}(x) + A_{n-5}^{(5)}(x)\Big) + O\Big(\frac{1}{\lambda^{n+1}}\Big)\Big], \quad (8)$$

And

$$y_k^{(6)}(x,\lambda) = -\lambda^6 e^{i\lambda w_k x}\Big[A_0(x) + \frac{1}{\lambda}\Big(A_1(x) - 6iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(A_2(x) - 6iw_k^5 A_1'(x) - 15w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(A_3(x) - 6iw_k^5 A_2'(x) - 15w_k^4 A_1''(x) + 20iw_k^3 A_0'''(x)\Big) + \frac{1}{\lambda^4}\Big(A_4(x) - 6iw_k^5 A_3'(x) - 15w_k^4 A_2''(x) + 20iw_k^3 A_1'''(x) + 15w_k^2 A_0^{(4)}(x)\Big) + \frac{1}{\lambda^5}\Big(A_5(x) - 6iw_k^5 A_4'(x) - 15w_k^4 A_3''(x) + 20iw_k^3 A_2'''(x) + 15w_k^2 A_1^{(4)}(x) - 6iw_k A_0^{(5)}(x)\Big) + \frac{1}{\lambda^6}\Big(A_6(x) - 6iw_k^5 A_5'(x) - 15w_k^4 A_4''(x) + 20iw_k^3 A_3'''(x) + 15w_k^2 A_2^{(4)}(x) - 6iw_k A_1^{(5)}(x) + A_0^{(6)}(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(A_n(x) - 6iw_k^5 A_{n-1}'(x) - 15w_k^4 A_{n-2}''(x) + 20iw_k^3 A_{n-3}'''(x) + 15w_k^2 A_{n-4}^{(4)}(x) - 6iw_k A_{n-5}^{(5)}(x) +$$

$A_{n-6}^{(6)}(x)\Big) + O\left(\frac{1}{\lambda^{n+1}}\right)\Big],$ (9)

Putting $y_k, y_k^{(6)}$ in $-y^{(6)}(x) + q(x)y = \lambda^6 y(x)$, then we get:

$-\lambda^6 e^{i\lambda w_k x}\Big[\frac{1}{\lambda}\Big(-6iw_k^5 A_0'(x)\Big) + \frac{1}{\lambda^2}\Big(-6iw_k^5 A_1'(x) - 15w_k^4 A_0''(x)\Big) + \frac{1}{\lambda^3}\Big(-6iw_k^5 A_2'(x) - 15w_k^4 A_1''(x) + 20iw_k^3 A_0'''(x)\Big) + \frac{1}{\lambda^4}\Big(-6iw_k^5 A_3'(x) - 15w_k^4 A_2''(x) + 20iw_k^3 A_1'''(x) + 15w_k^2 A_0^{(4)}(x)\Big) + \frac{1}{\lambda^5}\Big(-6iw_k^5 A_4'(x) - 15w_k^4 A_3''(x) + 20iw_k^3 A_2'''(x) + 15w_k^2 A_1^{(4)}(x) - 6iw_k A_0^{(5)}(x)\Big) + \frac{1}{\lambda^6}\Big(-6iw_k^5 A_5'(x) - 15w_k^4 A_4''(x) + 20iw_k^3 A_3'''(x) + 15w_k^2 A_2^{(4)}(x) - 6iw_k A_1^{(5)}(x) + A_0^{(6)}(x) + q(x)A_0(x)\Big) + \frac{1}{\lambda^7}\Big(-6iw_k^5 A_6'(x) - 15w_k^4 A_5''(x) + 20iw_k^3 A_4'''(x) + 15w_k^2 A_3^{(4)}(x) - 6iw_k A_2^{(5)}(x) + A_1^{(6)}(x) + q(x)A_1(x)\Big) + \frac{1}{\lambda^8}\Big(-6iw_k^5 A_7'(x) - 15w_k^4 A_6''(x) + 20iw_k^3 A_5'''(x) + 15w_k^2 A_4^{(4)}(x) - 6iw_k A_3^{(5)}(x) + A_2^{(6)}(x) + q(x)A_2(x)\Big) + \cdots + \frac{1}{\lambda^n}\Big(-6iw_k^5 A_{n-1}'(x) - 15w_k^4 A_{n-2}''(x) + 20iw_k^3 A_{n-3}'''(x) + 15w_k^2 A_{n-4}^{(4)}(x) - 6iw_k A_{n-5}^{(5)}(x) + A_{n-6}^{(6)}(x) + q(x)A_{n-6}(x)\Big) + O\left(\frac{1}{\lambda^{n+1}}\right)\Big] = 0.$

By equating the coefficients of the same power of $\frac{1}{\lambda}$. we get the following relation for $A_i(x), i = 1,2,\dots,n$:

$-6iw_k^5 A_0'(x) = 0,$

$-6iw_k^5 A_1'(x) - 15w_k^4 A_0''(x) = 0,$

$-6iw_k^5 A_2'(x) - 15w_k^4 A_1''(x) + 20iw_k^3 A_0'''(x) = 0,$

$-6iw_k^5 A_3'(x) - 15w_k^4 A_2''(x) + 20iw_k^3 A_1'''(x) + 15w_k^2 A_0^{(4)}(x) = 0,$

$-6iw_k^5 A_4'(x) - 15w_k^4 A_3''(x) + 20iw_k^3 A_2'''(x) + 15w_k^2 A_1^{(4)}(x) - 6iw_k A_0^{(5)}(x) = 0,$

$-6iw_k^5 A_5'(x) - 15w_k^4 A_4''(x) + 20iw_k^3 A_3'''(x) + 15w_k^2 A_2^{(4)}(x) - 6iw_k A_1^{(5)}(x) + q(x)A_0(x) = 0,$

$-6iw_k^5 A_6'(x) - 15w_k^4 A_5''(x) + 20iw_k^3 A_4'''(x) + 15w_k^2 A_3^{(4)}(x) - 6iw_k A_2^{(5)}(x) + A_1^{(6)}(x) + q(x)A_1(x) = 0,$

$-6iw_k^5 A_7'(x) - 15w_k^4 A_6''(x) + 20iw_k^3 A_5'''(x) + 15w_k^2 A_4^{(4)}(x) - 6iw_k A_3^{(5)}(x) + A_2^{(6)}(x) + q(x)A_2(x) = 0$

$\vdots$

$-6iw_k^5 A_{n-1}'(x) - 15w_k^4 A_{n-2}''(x) + 20iw_k^3 A_{n-3}'''(x) + 15w_k^2 A_{n-4}^{(4)}(x) - 6iw_k A_{n-5}^{(5)}(x) + A_{n-6}^{(6)}(x) + q(x)A_{n-6}(x) = 0.$

By solving the above equations we get :

From Eq.(1) $A_0'(x) = 0$ then $A_0(x) = 1$ put in Eq. (2) we get :

$A_{0,k}(x) = A_{1,k}(x) = A_{2,k}(x) = A_{3,k}(x) = A_{4,k}(x) = 1$

$A_{5.k}(x) = -\frac{iw_k}{6}\int_0^x q(t)A_{0,k}(t)dt,$

$A_{6,k}(x) = -\frac{iw_k}{6}\int_0^x \Big(-15w_k^4 A_{5,k}''(t) + q(t)A_{1,k}(t)\Big)dt,$

$A_{7,k}(x) = -\frac{iw_k}{6}\int_0^x \Big(-15w_k^4 A_{6,k}''(t) + 20iw_k^3 A_{5,k}'''(t) + q(t)A_{2,k}(t)\Big)dt.$

$$A_{8,k}(x) = -\frac{iw_k}{6}\int_0^x \left(-15w_k^4 A''_{7,k}(t) + 20iw_k^3 A'''_{6,k}(t) + 15w_k^2 A^{(4)}_{5,k}(t) + q(t)A_{3,k}(t)\right)dt.$$

And hence for integer $n \geq 9$ we get that :

$$A_{n.k}(x) = -\frac{iw_k}{6}\int_0^x \left(-15w_k^4 A''_{n-1,k}(t) + 20iw_k^3 A'''_{n-2,k}(t) + 15w_k^2 A^{(4)}_{n-3,k}(t) + A^{(5)}_{n-4,k}(t) + A^{(6)}_{n-5,k}(t) + q(t)A_{n-5,k}(t)\right)dt.$$

And then

$$A_{1sk} = A_{1k}(x) - \binom{S}{1} iw_k^5 A'_{0k}(x),$$

$$A_{2sk} = A_{2k}(x) - \binom{S}{1} iw_k^5 A'_{1k}(x) - \binom{S}{2} w_k^4 A''_{0k}(x),$$

$$A_{3sk} = A_{3k}(x) - \binom{S}{1} iw_k^5 A'_{2k}(x) - \binom{S}{2} w_k^4 A''_{1k}(x) + \binom{S}{3} w_k^3 A'''_{0k}(x),$$

$$A_{4sk} = A_{4k}(x) - \binom{S}{1} iw_k^5 A'_{3k}(x) - \binom{S}{2} w_k^4 A''_{2k}(x) + \binom{S}{3} iw_k^3 A'''_{1k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{0k}(x),$$

$$A_{5sk} = A_{5k}(x) - \binom{S}{1} iw_k^5 A'_{4k}(x) - \binom{S}{2} w_k^4 A''_{3k}(x) + \binom{S}{3} w_k^3 A'''_{2k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{1k}(x) - \binom{S}{5} iw_k A^{(5)}_{0k}(x),$$

$$A_{6sk} = A_{6k}(x) - \binom{S}{1} iw_k^5 A'_{5k}(x) - \binom{S}{2} w_k^4 A''_{4k}(x) + \binom{S}{3} iw_k^3 A'''_{3k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{2k}(x) - \binom{S}{5} iw_k A^{(5)}_{1k}(x) + \binom{S}{6} A^{(6)}_{0k}(x),$$

$$A_{7sk} = A_{7k}(x) - \binom{S}{1} iw_k^5 A'_{6k}(x) - \binom{S}{2} w_k^4 A''_{5k}(x) + \binom{S}{3} iw_k^3 A'''_{4k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{3k}(x) - \binom{S}{5} iw_k A^{(5)}_{2k}(x) + \binom{S}{7} A^{(7)}_{0k}(x),$$

⋮

And so on for $n \geq 7$ we have:

$$A_{nsk} = A_{n,k}(x) - \binom{S}{1} iw_k^5 A'_{n-1,k}(x) - \binom{S}{2} w_k^4 A''_{n-2,k}(x) + \binom{S}{3} iw_k^3 A'''_{n-3,k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{n-4,k}(x) - \binom{S}{5} iw_k A^{(5)}_{n-5,k}(x) + \binom{S}{6} A^{(6)}_{n-6,k}(x).$$

But since $A_{0k}(x), A_{1k}(x)A_{2k}(x), A_{3k}(x)$ and $A_{4k}(x)$ are constants, so all derivatives are vanish. Then we can write:

$A_{1sk} = A_{1k}(x), A_{2sk} = A_{2k}(x), A_{3sk} = A_{3k}(x), A_{4sk} = A_{4k}(x),$

$$A_{5sk} = A_{5k}(x),$$

$$A_{6sk} = A_{6k}(x) - \binom{S}{1} iw_k^5 A'_{5k}(x),$$

$$A_{7sk} = A_{7k}(x) - \binom{S}{1} iw_k^5 A'_{6k}(x) - \binom{S}{2} w_k^4 A''_{5k}(x),$$

⋮

And so on for $n \geq 7$ we have:

$$A_{nsk} = A_{n,k}(x) - \binom{S}{1} iw_k^5 A'_{n-1,k}(x) - \binom{S}{2} w_k^4 A''_{n-2,k}(x) + \binom{S}{3} iw_k^3 A'''_{n-3,k}(x) + \binom{S}{4} w_k^2 A^{(4)}_{n-4,k}(x) - \binom{S}{5} iw_k A^{(5)}_{n-5,k}(x) + \binom{S}{6} A^{(6)}_{n-6,k}(x).$$

Hence we can write the four linearly independent solution and their derivatives of the differential equation of the form

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k)^s e^{i\lambda w_k x}\left[A_{0k}(x) + \frac{A_{1sk}}{\lambda} + \frac{A_{2sk}}{\lambda^2} + \frac{A_{3sk}}{\lambda^3} + \frac{A_{4sk}}{\lambda^4} + \frac{A_{5sk}}{\lambda^5} + \frac{A_{6sk}}{\lambda^6} + \cdots + \frac{A_{nsk}}{\lambda^n} + O\left(\frac{1}{\lambda^{n+1}}\right)\right].$$

Theorem 2: Consider the boundary value problem(1.1), (1.2), where $q(x)$ is smooth function, such that satisfies the conditions $q'(a) = 0\,, q'(0) = 0$ $\,, \int_0^a q(x)dx = 0$, and $q(a) \neq 0$, then for $\lambda \in T_0$, the asymptotic formulas for eigenvalues of the problem for sufficiently large $|m|$,has the following forms

$$\lambda_{0,m} = \left(\frac{m\pi(1+i\sqrt{3})}{a} - 1 + 214\frac{(18q(a)-54iq(a))}{(\frac{2m\pi}{a})^6} + O\left(\frac{1}{m^8}\right)\right)^6.$$

For $m = N, N+1, N+2, \ldots$Where N is a large integer.

Proof: If we choose five terms of $y_k^{(s)}(x,\lambda)$ in Theorem 1 then:

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k')^s e^{i\lambda w_k x}\left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + \frac{A_{5sk}(x)}{\lambda^5} + O\left(\frac{1}{\lambda^6}\right)\right]$$

For, $s = 1,2,3,4,5$, $k = 0,1,2,3,4,5$. We have :

$A_{0sk} = A_{0k}(x)\,,\ A_{1sk} = A_{1k}(x)\,,\ A_{2sk} = A_{2k}(x),\ A_{3sk} = A_{3k}(x),$

$A_{4sk} = A_{4k}(x)\,, \qquad A_{5sk} = A_{5k}(x)$

$A_{6sk} = A_{6k}(x) - \binom{s}{1} i{w_k'}^5 A_{5k}'(x).$

$A_{7sk} = A_{7k}(x) - \binom{s}{1} i{w_k'}^5 A_{6k}'(x) - \binom{s}{2} {w_k'}^4 A_{5k}''(x),$

And

$A_{0k}(x) = 1,\ A_{1k}(x) = 0,\ A_{2k}(x) = 0, A_{3k}(x) = 0,\ A_{4k}(x) = 0,$

$A_{5k}(x) = -\frac{iw_k'}{6}\int_0^x q(t)A_{0k}(t)dt,$

$A_{6k}(x) = -\frac{iw_k'}{6}\int_0^x \left(-15{w_k'}^4 A_{5k}''(t) + q(t)A_{1k}(t)\right)dt,$

$A_{7k}(x) = -\frac{iw_k'}{6}\int_0^x \left(-15{w_k'}^4 A_{6k}''(t) + 20i{w_k'}^3 A_{3k}'''(t) + q(t)A_{2k}(t)\right)dt,$

Then ,

$A_{0sk} = 1\,, A_{1sk} = 0\,, A_{2sk} = 0, A_{3sk} = 0\,, A_{4sk} = 0,$

$A_{5sk} = -\frac{iw_k'}{6}\int_0^x q(t)dt, A_{60k} = \frac{5}{12}[q(x) - q(0)],$

$A_{61k} = \left[\frac{3}{12}q(x) - \frac{3}{8}q(0)\right], A_{62k} = \left[\frac{1}{12}q(x) - \frac{5}{12}q(0)\right],$

$A_{63k} = \left[-\frac{1}{12}q(x) - \frac{5}{12}q(0)\right], A_{64k} = \left[-\frac{3}{12}q(x) - \frac{5}{12}q(0)\right]$

$A_{65k} = \left[-\frac{5}{12}q(x) - \frac{5}{12}q(0)\right]$

$A_{70k}=\frac{5iw_k'^5}{72}[q'(x)-q'(0)]$, $A_{71k}=\left[-\frac{25i}{72}w_k'^5q'(x)-\frac{5i}{72}w_k'^5q'(0)\right]$,

$A_{72k}=\left[-\frac{43i}{72}w_k'^5q'(x)-\frac{5i}{72}w_k'^5q'(0)\right]$,

$A_{73k}=\left[-\frac{49i}{72}w_k'^5q'(x)-\frac{5i}{72}w_k'^5q'(0)\right]$,

$A_{74k}=\left[-\frac{43i}{72}w_k'^5q'(x)-\frac{5i}{72}w_k'^5q'(0)\right]$

$A_{75k}=\left[-\frac{25i}{72}w_k'^5q'(x)-\frac{5i}{72}w_k'^5q'(0)\right]$

And to find the boundary conditions $U_j(y_k)$ for $k,j=0,1,2,3,4,5$. Where $U_0(y)=y(0)=0, U_1(y)=y'(0)=0, U_2(y)=y''(0)=0, U_j(y)=\sum_{k=1}^{6}\left(iw_j\lambda\right)^{6-k}y^{(k-1)}(a,\lambda)=0$

$j=3,4,5$. where, $w_k=\sqrt[6]{1}=e^{\frac{2\pi k}{6}i}=e^{\frac{\pi k}{3}i}, k=0,1,2,3,4,5$. And $q(x)$ is smooth function.

Now $w_0=1$and $w_1=\frac{1}{2}+i\frac{\sqrt{3}}{2}, w_2=\frac{-1}{2}+i\frac{\sqrt{3}}{2}, w_3=-1, w_4=\frac{-1}{2}-i\frac{\sqrt{3}}{2}, w_5=\frac{1}{2}-i\frac{\sqrt{3}}{2}$. And w_j' are the w_j which numbering so that satisfy the inequality:

$$Re(i\lambda w_0')\le Re(i\lambda w_1')\le Re(i\lambda w_2')\le Re(i\lambda w_3')\le Re(i\lambda w_4')\le Re(i\lambda w_5') \quad (10)$$

We can easily find out the form of each boundary conditions up to order six in each sectors:

$$U_0(y_k)=\left[1+O\left(\frac{1}{\lambda^8}\right)\right] \quad (11)$$

$$U_1(y_k)=i\lambda w_k'\left[1-\frac{1}{6}\frac{q(0)}{\lambda^6}-\frac{5iw_k'^5q'(0)}{12\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right] \quad (12)$$

$$U_2(y_k)=-\lambda^2w_k'^2\left[1+\frac{1}{6}\frac{q(0)}{\lambda^6}-\frac{5iw_k'^5q'(0)}{12\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right] \quad (13)$$

$$\begin{aligned}U_j(y_k)=i\lambda^5e^{i\lambda w_k a}\Bigg[&(w_k')^5\left(\left[1+\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[-\frac{5}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[-\frac{25i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right]\right)+\\
&(w_j)(w_k')^4\left(\left[1+\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[-\frac{3}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[-\frac{43i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right]\right)+\\
&(w_j)^2(w_k')^3\left(\left[1+\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[\frac{-1}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[-\frac{49i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right]\right)+\\
&(w_j)^3(w_k')^2\left(\left[1+\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[\frac{1}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[-\frac{43i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right]\right)+\\
&(w_j)^4(w_k')\left(\left[1+\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[\frac{3}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[-\frac{25i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\right]\right)+(w_j)^5\Bigg(\Bigg[1+\\
&\frac{-\frac{iw_k'}{6}\int_0^a q(t)dt}{\lambda^5}+\frac{\left[\frac{5}{12}q(a)-\frac{5}{12}q(0)\right]}{\lambda^6}+\frac{\left[\frac{5i}{72}w_k'^5\left(q'(a)\right)-\frac{5i}{72}w_k'^5q'(0)\right]}{\lambda^7}+O\left(\frac{1}{\lambda^8}\right)\Bigg]\Bigg)\Bigg] \quad (14)\end{aligned}$$

For $j=3,4,5$

If $\lambda \in T_0$, then
$w'_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, w'_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, w'_2 = 1, w'_3 = -1, w'_4 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, w'_5 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.$

And by our assumption we have $q'(a) = 0\ and\ q'(0) = 0\ and\ \int_0^a q(x)dx = 0$, then after computation from equation (11, 14) we get

$$U_0(y_k) = \left[1 + O\left(\frac{1}{\lambda^8}\right)\right] = A,$$

$$U_1(y_k) = i\lambda w'_k\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} - \frac{5iw'^{5}_k q'(0)}{12\lambda^7} + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_1(y_0) = i\lambda(\frac{1}{2} + \frac{i\sqrt{3}}{2})\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = i\lambda(\frac{1}{2} + \frac{i\sqrt{3}}{2})A,$$

$$U_1(y_1) = i\lambda\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right)\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^6}\right)\right] = i\lambda\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right)A$$

$$U_1(y_2) = i\lambda\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^6}\right)\right] = i\lambda A, U_1(y_3) = -i\lambda\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^6}\right)\right] = -i\lambda A,$$

$$U_1(y_4) = i\lambda(\frac{1}{2} - \frac{i\sqrt{3}}{2})\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = i\lambda(\frac{1}{2} - \frac{i\sqrt{3}}{2})A,$$

$$U_1(y_5) = i\lambda\left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right)\left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = i\lambda(\frac{-1}{2} - \frac{i\sqrt{3}}{2})A,$$

Where $A = \left[1 - \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right]$.

$$U_2(y_k) = -\lambda^2 w'^{2}_k\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} - \frac{5iw'^{5}_k q'(0)}{12\lambda^7} + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_2(y_0) = -\lambda^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 B,$$

$$U_2(y_1) = -\lambda^2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 B,$$

$$U_2(y_2) = -\lambda^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2 B,$$

$$U_2(y_3) = -\lambda^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2 B$$

$$U_2(y_4) = -\lambda^2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 B$$

$$U_2(y_5) = -\lambda^2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2\left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] = -\lambda^2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 B$$

Where $B = \left[1 + \frac{1}{6}\frac{q(0)}{\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right]$

$$U_3(y_0) = i\lambda^5 e^{i\lambda w'_0 a}\left[\frac{1}{12\lambda^6}\left(-6q(a) + 2i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_3(y_1) = i\lambda^5 e^{i\lambda w_1' a}\left[\frac{1}{12\lambda^6}\left(-6q(a) + 6i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_3(y_2) = i\lambda^5 e^{i\lambda w_2' a}\left[\frac{-6}{12\lambda^6}q(a) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_3(y_3) = i\lambda^5 e^{i\lambda w_3' a}\left[-6 + \frac{30}{12\lambda^6}q(0) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_3(y_4) = i\lambda^5 e^{i\lambda w_4' a}\left[\frac{1}{12\lambda^6}\left(-6q(a) - 2i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_3(y_5) = i\lambda^5 e^{i\lambda w_5' a}\left[\frac{1}{12\lambda^6}\left(6q(a) - 6i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_0) = i\lambda^5 e^{i\lambda w_0' a}\left[\frac{1}{12\lambda^6}\left(6q(a) + 6i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_1) = i\lambda^5 e^{i\lambda w_1' a}\left[\frac{4i\sqrt{3}}{12\lambda^6}q(a) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_2) = i\lambda^5 e^{i\lambda w_2' a}\left[\frac{1}{12\lambda^6}\left(-6q(a) + 2i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_3) = i\lambda^5 e^{i\lambda w_3' a}\left[\frac{1}{12\lambda^6}\left(6q(a) + 6i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_4) = i\lambda^5 e^{i\lambda w_4' a}\left[\frac{-12}{12\lambda^6}q(a) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_4(y_5) = i\lambda^5 e^{i\lambda w_5' a}\left[-3 + 3i\sqrt{3} + \frac{1}{12\lambda^6}\left(15q(0) - 15i\sqrt{3}q(0)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_0) = i\lambda^5 e^{i\lambda w_0' a}\left[\frac{4i\sqrt{3}}{12\lambda^6}q(a) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_1) = i\lambda^5 e^{i\lambda w_1' a}\left[\frac{1}{12\lambda^6}\left(3q(a) + 3i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_2) = i\lambda^5 e^{i\lambda w_2' a}\left[\frac{1}{12\lambda^6}\left(-6q(a) + 6i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_3) = i\lambda^5 e^{i\lambda w_3' a}\left[\frac{1}{12\lambda^6}\left(6q(a) + 2i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_4) = i\lambda^5 e^{i\lambda w_4' a}\left[3 + 3i\sqrt{3} + \frac{1}{12\lambda^6}\left(-15q(a) - 15i\sqrt{3}q(a)\right) + O\left(\frac{1}{\lambda^8}\right)\right]$$

$$U_5(y_5) = i\lambda^5 e^{i\lambda w_5' a}\left[\frac{12}{12\lambda^6}q(a) + O\left(\frac{1}{\lambda^8}\right)\right]$$

We want to find $\Delta(\lambda)$, in T_0

$$\Delta(\lambda) = \begin{vmatrix} U_0(y_0) & U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) & U_0(y_5) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) \end{vmatrix}$$

$$= \lambda^{18}ABC \begin{vmatrix} U_0(y_0) & U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) & U_0(y_5) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) \end{vmatrix}$$

Then by Laplace expansion theorem for determinant we get as we see in [7], [8], we can reduce $\Delta(\lambda)$ to

$$\Delta(\lambda) = i^3\lambda^{18}ABC\left[e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}\left[216-\frac{1}{\lambda^6}\left(180q(0)+90q(a)+O\left(\frac{1}{\lambda^8}\right)\right)\right]+\frac{18(\sqrt{3}-3i)}{\lambda^6}\left(q(a)+i\sqrt{3}q(a)\right)+O\left(\frac{1}{\lambda^8}\right)\right]$$

Clearly we known from [8], the eigenvalues of the given problem are the zeros of $\Delta(\lambda)$. If $\Delta(\lambda) = 0$ for sufficiently large $|\lambda|$, then

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}\left[216-\frac{1}{\lambda^6}\left(180q\left(0+90q(a)\right)+O\left(\frac{1}{\lambda^8}\right)\right)\right]+\frac{18(\sqrt{3}-3i)}{\lambda^6}\left(q(a)+i\sqrt{3}q(a)\right)+O\left(\frac{1}{\lambda^8}\right)=0$$

(15)

Then

$$216-\frac{1}{\lambda^6}\left(180q(0)+90q(a)\right)+O\left(\frac{1}{\lambda^8}\right)\Big) = 1-\left(-215+\frac{(180q(0)+90q(a))}{\lambda^6}+O\left(\frac{1}{\lambda^6}\right)\right)$$

And then

$$\left[216-\frac{1}{\lambda^6}\left(180q(0)+90q(a)+O\left(\frac{1}{\lambda^8}\right)\right)\right]^{-1} = 1+\left(-215+\frac{(180q(0)+90q(a))}{\lambda^6}+O\left(\frac{1}{\lambda^6}\right)\right) =$$

$$-214+\frac{180q(0)+90q(a)}{\lambda^6}+O\left(\frac{1}{\lambda^6}\right),\ \{\text{Since } \frac{1}{1-x}=1+x+\cdots\ \}$$

And

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}+\left[-214+\frac{180q(0)+90q(a)}{\lambda^6}+O\left(\frac{1}{\lambda^6}\right)\right]\left[\frac{18(\sqrt{3}-3i)}{\lambda^6}\left(q(a)+i\sqrt{3}q(a)\right)+O\left(\frac{1}{\lambda^8}\right)\right]=0$$

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}+\left[-214+\frac{180q(0)+90q(a)}{\lambda^6}+O\left(\frac{1}{\lambda^8}\right)\right]\left[\frac{1}{\lambda^6}\left(18\sqrt{3}q(a)-54i\sqrt{3}q(a)\right)+O\left(\frac{1}{\lambda^8}\right)\right]=0$$

So from equation (15) we find out:

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}-\frac{214}{\lambda^6}\left(18\sqrt{3}q(a)-54i\sqrt{3}q(a)\right)+O\left(\frac{1}{\lambda^8}\right)=0$$

Then,

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\sqrt{3}=\frac{214\sqrt{3}}{\lambda^6}\left(18q(a)-54iq(a)\right)+O\left(\frac{1}{\lambda^8}\right)$$

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}=\frac{214}{\lambda^6}\left(18q(a)-54iq(a)\right)+O\left(\frac{1}{\lambda^8}\right)$$

$$e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}-1=-1+\frac{214}{\lambda^6}\left(18q(a)-54iq(a)\right)+O\left(\frac{1}{\lambda^8}\right)$$

Then according to[2] by using Rouche's theorem we can solve it and we get:

$$i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a=2m\pi i\rightarrow\lambda=\frac{2m\pi}{\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}$$

$\lambda=\frac{2m\pi}{\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}-1+214\frac{(18q(a)-54iq(a)}{\left(\frac{2m\pi}{\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\right)^6}+O\left(\frac{1}{m^8}\right)$, for $m=N,N+1,N+2,\dots$

Where N is a large integer.

Hence

$$\lambda=\frac{4m\pi}{(1-i\sqrt{3})a}-1+214\frac{(18q(a)-54iq(a))}{(\frac{2m\pi}{a})^6}+O\left(\frac{1}{m^8}\right).$$

$$\lambda=\frac{m\pi\left(1+i\sqrt{3}\right)}{a}-1+214\frac{(18q(a)-54iq(a))}{(\frac{2m\pi}{a})^6}+O\left(\frac{1}{m^8}\right)$$

And we know that the eigen-values of the problem are $\lambda_{0,m}=\lambda^6$

That is ,

$$\lambda_{0,m}=\left(\frac{m\pi(1+i\sqrt{3})}{a}-1+214\frac{(18q(a)-54iq(a))}{(\frac{2m\pi}{a})^6}+O\left(\frac{1}{m^8}\right)\right)^6.$$

$,for\ m=N,N+1,N+2,\dots$

Where N is a large integer.

We can use binomial coefficients theorem and properties of big (O) notation to get the expression of the eigenvalue in the sector T_0.

Theorem 3 : Asymptotic formulas of the Eigen-function of the boundary value problem (1)-(2) corresponding to $\lambda_{j,m}$, $\lambda_{\bar{j},m}$, for$j=0{:}\,5$ has the following forms:

$$y_{0,m}(x,\lambda)=-i\sqrt{3}e^{i\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}+e^{i\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}-e^{i\lambda x}+O\left(\frac{1}{\lambda^5}\right)\ \ \lambda\in T_k$$

$k,j=0{:}\,5,\ \ for\ m=N,N+1,N+2,\dots$

Where N is a large integer.

Proof: If we choose five terms of $y_k^{(s)}(x,\lambda)$ in the above equation then:

$$y_k^{(s)}(x,\lambda) = (i\lambda w_k')^s e^{i\lambda w_k' x}\left[A_{0sk}(x) + \frac{A_{1sk}(x)}{\lambda} + \frac{A_{2sk}(x)}{\lambda^2} + \frac{A_{3sk}(x)}{\lambda^3} + \frac{A_{4sk}(x)}{\lambda^4} + O\left(\frac{1}{\lambda^3}\right)\right]$$

For $s = 0,1,2,3,4,5$, $k = 0,1,2,3,4,5$, We have:

$$A_{0sk} = A_{0k}, A_{1sk} = A_{1k}, A_{2sk} = A_{2k}, A_{3sk} = A_{3k}, A_{4sk} = A_{4k}$$

$$A_{0sk} = 1\,, \qquad A_{1sk} = 0\,, \qquad A_{2sk} = 0, A_{3sk} = 0, A_{4sk} = 0$$

$$A_{5k}(x) = \frac{-iw_k'}{6}\int_0^x q(t)A_{0k}(t)dt$$

And to finding the boundary conditions $U_j(y_k)$ for $k = 0,1,2,3,4,5,$ $j = 1,2,3,4,5$ Up to order $O\left(\frac{1}{\lambda^5}\right)$ and satisfy $\int_0^a q(t)dt = 0$, and If $\lambda \in T_0$, then $w_0' = \frac{1}{2} + i\frac{\sqrt{3}}{2}, w_1' = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, w_2' = 1 =, w_3' = -1, w_4' = \frac{1}{2} - i\frac{\sqrt{3}}{2}, w_5' = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

Now we can easily find

$$U_1(y_k) = y_k'(0) = i\lambda w_k' e^{i\lambda w_k' 0}\left[A_{01k}(0) + \frac{A_{11k}(0)}{\lambda} + \frac{A_{21k}(0)}{\lambda^2} + \frac{A_{31k}(0)}{\lambda^3} + \frac{A_{41k}(0)}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right)\right]$$

$U_1(y_k) = i\lambda w_k'\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$, for $k = 0,1,2,3,4,5$

$U_1(y_0) = i\lambda\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$, $U_1(y_1) = i\lambda\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$,

$U_1(y_2) = i\lambda\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$, $U_1(y_3) = -i\lambda\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$,

$$U_1(y_4) = i\lambda\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_1(y_5) = i\lambda\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2(y_k) = (i\lambda w_k')^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right] = -\lambda^2 {w_k'}^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2\,(y_0) = -\lambda^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right],$$

$$U_2\,(y_1) = -\lambda^2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2\,(y_2) = -\lambda^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2\,(y_3) = -\lambda^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2\,(y_4) = -\lambda^2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_2\,(y_5) = -\lambda^2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2\left[1 + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_j\,(y_k) = \sum_{l=1}^{6} \left(iw_j\lambda\right)^{l-1} y^{(6-l)}(a,\lambda)$$

$$U_j\,(y_k) = i\lambda^5 e^{i\lambda w'_k a}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]\left[(w'_k)^5 + w_j(w'_k)^4 + \left(w_j\right)^2(w'_k)^3 + \left(w_j\right)^3(w'_k)^2 + \left(w_j\right)^4(w'_k) + \left(w_j\right)^5\right],$$

for $k = 0:5, j = 3,4,5$

$$U_3\,(y_0) = U_3\,(y_1) = U_3\,(y_2) = 0$$

$$U_3\,(y_3) = -6i\lambda^5 e^{-i\lambda a}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

$$U_3\,(y_4) = U_3\,(y_5) = U_4\,(y_0) = U_4\,(y_1) = U_4\,(y_2) = U_4\,(y_3) = U_4\,(y_4) = 0$$

$$U_4\,(y_5) = i\lambda^5 e^{i\lambda\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\left[-3+3i\sqrt{3}\right]\left[1+O\left(\frac{1}{\lambda^5}\right)\right],$$

$$U_5\,(y_0) = U_5\,(y_1) = U_5\,(y_2) = U_5\,(y_3) = 0$$

$$U_5\,(y_4) = i\lambda^5 e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}\left[3+3i\sqrt{3}\right]\left[1+O\left(\frac{1}{\lambda^5}\right)\right],$$

$$U_5\,(y_5) = 0$$

$$y_0(x,\lambda) = e^{i\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right],$$

$$y_1(x,\lambda) = e^{i\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

$$y_2(x,\lambda) = e^{i\lambda x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

$$y_3(x,\lambda) = e^{-i\lambda x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

$$y_4(x,\lambda) = e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

$$y_5(x,\lambda) = e^{i\lambda\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)x}\left[1+O\left(\frac{1}{\lambda^5}\right)\right]$$

As we see in [9] we can write the eigenfunction in T_0 as follows

$$y_{0,m}(x,\lambda) = \frac{6i^4\lambda^{18}\left(-3+3i\sqrt{3}\right)\left(3+3i\sqrt{3}\right)}{6i^4\lambda^{18}\left(-3+3i\sqrt{3}\right)\left(3+3i\sqrt{3}\right)} e^{i\lambda a} e^{i\lambda\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a} e^{i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a} e^{-i\lambda\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a} e^{-i\lambda\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)a}$$

$$\left[\begin{vmatrix} y_0(x,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) & y_5(x,\lambda) \\ U_1(y_0) & U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) & U_1(y_5) \\ U_2(y_0) & U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) & U_2(y_5) \\ U_3(y_0) & U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) & U_3(y_5) \\ U_4(y_0) & U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) & U_4(y_5) \\ U_5(y_0) & U_5(y_1) & U_5(y_2) & U_5(y_3) & U_5(y_4) & U_5(y_5) \end{vmatrix} + O\left(\frac{1}{\lambda^5}\right)\right]$$

$$= \Bigg[e^{i\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^2\right)-e^{i\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\right)$$

$$+e^{i\lambda x}\left(-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)\Bigg]*[-1(-1)-0+0]$$

$$+O\left(\frac{1}{\lambda^5}\right)$$

$$= -i\sqrt{3}e^{i\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}+e^{i\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}-e^{i\lambda x}+O\left(\frac{1}{\lambda^5}\right)$$

So $y_{0,m}(x,\lambda) = -i\sqrt{3}e^{i\lambda\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}+e^{i\lambda\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)x}-e^{i\lambda x}+O\left(\frac{1}{\lambda^5}\right)$

Since in T_0 $\lambda = \lambda_{0,m}$, where $\lambda_{0,m} = -\frac{1}{2a}\left[2m\pi + i + 28\frac{q(a)}{\left(\frac{1}{a}m\pi\right)^4}\right] + O\left(\frac{1}{m^6}\right)$

So

For $m = N, N+1, N+2, \dots$. Where N is a large integer.

Conclusion

In this paper, we obtain asymptotic expressions for the sixth linearly independent solutions, and also the asymptotic formulas for the eigenvalues and eigenfunctions of the boundary value problem (1)-(2) were demonstrated.

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